MA40050: Numerical Optimisation & Large–Scale Systems

Problem Sheet 6

Instructions: Submit solutions to Question 1 and 4 by **Thursday, 30th April, 12.15pm**
As a single file through Moodle (either PDF or ZIP of photos/scans).
Name your file '(given name) (family name) PS6'.

1. **Quadratic Program.** Let $b \in \mathbb{R}^M$ and $g \in \mathbb{R}^N$, $M \leq N$. Suppose $A \in \mathbb{R}^{M \times N}$ has full rank and $H = H^T \in \mathbb{R}^{N \times N}$ is positive definite on $\ker(A)$, i.e.

$$x^T H x > 0, \quad \text{for all } x \neq 0 \text{ with } Ax = 0.$$

   (a) Show that the KKT conditions for the *quadratic program*

$$\min_{x \in \mathbb{R}^N} \tfrac{1}{2} x^T H x - x^T g \quad \text{subject to} \quad Ax = b, \tag{1}$$

   can be written (in block matrix form) as

$$\begin{pmatrix} H & -A^T \\ -A & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} g \\ -b \end{pmatrix}. \tag{2}$$

   (b) Show that the *KKT matrix* in (2) is invertible and deduce that (1) has a unique solution.

2. **Algorithm 8.1 – Newton's Method for Equality Constraints.** Let $f \in \mathrm{C}^2(\mathbb{R}^N;\mathbb{R})$, $c \in \mathrm{C}^2(\mathbb{R}^N;\mathbb{R}^M)$ with $M = M_e \leq N$ (i.e., we only have equality constraints) such that $\nabla^2 f$ and $\nabla^2 c_j$, $j = 1, \ldots, M$, are Lipschitz continuous. Moreover, suppose that the pair (x_*, λ_*) satisfies the LICQ, the KKT conditions and the strong second-order optimality condition

$$d^T \nabla_x^2 \mathcal{L}(x_*, \lambda_*) d > 0 \qquad \forall d \in T_\Omega(x_*) = \mathcal{C}(x_*, \lambda_*).$$

   Prove that Newton's method for the nonlinear KKT system $\nabla_{x,\lambda} \mathcal{L}(x, \lambda) = 0$ is q-quadratically convergent, provided the initial guess (x_0, λ_0) is sufficiently close to (x_*, λ_*).

3. **Matlab.** Consider minimising the Rosenbrock function $f(x) := (1 - x_1)^2 + 10(x_2 - x_1^2)^2$ in $\Omega := \{x \in \mathbb{R}^2 : c_1(x) := 1 - x_1^2 - x_2^2 \geq 0\}$. Let $\mathcal{A}(x_*) = \{1\}$ and apply the `Matlab` implementation of Newton's method from **Problem Sheet 1** to solve $\nabla_{x,\lambda} \mathcal{L}(x, \lambda) = 0$ starting with $x_0 = (0.5, 0.5)^T$ and $\lambda_0 = 1$. What is the solution (x_*, λ_*)? Are the KKT conditions and the 2nd-order optimality condition satisfied at (x_*, λ_*)? What can you deduce about the minimiser of f in Ω? What happens if you change x_0?

   [You may assume that the minimiser of the unconstrained problem is $x_* = (1, 1)^T \notin \Omega$.]

P.T.O.

4. **Active Set Method.** Consider minimising $f(x) := x_1^2 + x_2^2$ in

$$\Omega := \{x \in \mathbb{R}^2 : c_1(x) := x_1^2 + (x_2 - 1)^2 - 1 \geq 0 \text{ and } c_2(x) := 1 - x_1^2 - (x_2 - 2)^2 \geq 0\}.$$

(a) Write down the KKT conditions for this problem and, using the active set method, find all the points (x_*, λ_*) that satisfy the KKT conditions.

(b) Verify that LICQ holds at each of the points and check the 2nd order optimality condition to decide which of the KKT points (x_*, λ_*) that you found is a local minimiser of f in Ω.