MA40050: Numerical Optimisation & Large–Scale Systems

Problem Sheet 2

Instructions: Hand in solutions to Questions 1 & 2 by **Thursday, 27th February, 12.15pm** (either in one of the lectures or to the pigeon hole in 4W Level 1).

1. Prove **Proposition 2.8**. Suppose x_* is a local minimizer of $f : \mathbb{R}^N \to \mathbb{R}$ in $\mathbb{R}^N$.

   (a) If f is differentiable at x_* then $\nabla f(x_*) = 0$.

   (b) If f is twice differentiable at x_* then $\nabla^2 f(x_*) \geq 0$.

2. **A Quadratic Optimisation Problem.** Let $A \in \mathbb{R}^{N\times N}$ be symmetric and positive definite (spd), $b \in \mathbb{R}^N$, $c \in \mathbb{R}$ and define $f(x) = \frac{1}{2}x^T Ax - b \cdot x + c$.

   (a) Show that $f \in \mathrm{C}^2(\mathbb{R}^N)$, that $\nabla f(x) = Ax - b$, and that $\nabla^2 f(x) = A$ for all $x \in \mathbb{R}^N$.

   (b) Deduce that f has a unique critical point x_* which is a strict local minimizer. Is it also a global minimizer?

3. Let $F(x) := x - \exp(-x)$, $x \in \mathbb{R}$, and let $x^0 = 1$.

   (a) Apply a few steps of fixpoint iteration with $G(x) := \exp(-x)$ to find approximations to the root of $F(x)$.

   (b) Apply a few steps of Newton's method to find approximations to the root of $F(x)$.

   Comment on the results.

4. **Behaviour of Newton's Method.**

   (a) Let $F(x) = x^k$, $k \geq 2$. Prove that, for any starting point $x^0 \neq 0$, Newton's method is well-defined and converges linearly to zero. Compute the convergence factor.

   (b) Give an example of a strictly increasing real function $F : \mathbb{R} \to \mathbb{R}$ with $F(0) = 0$, as well as a starting point x^0 so that Newton's method does not converge.
   [**Hint:** let F be skew-symmetric and try to create a cyclic behaviour.]

5. Implement **Algorithm 3.1**, Newton's Method, in `Matlab`, and then use it to find the minimum of the Rosenbrock function

$$f(x) = (1 - x_1)^2 + 100(x_2 - x_1^2)^2, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2,$$

   starting from $x^0 = (-2, 2)^T$. Use the `Matlab` function `eig()` to verify that the critical point you found is indeed a local minimizer.