MA40050: Numerical Optimisation & Large–Scale Systems

Review of Eigenvalues and Eigenvectors

For any symmetric $A \in \mathbb{R}^{N\times N}$ there exist eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N \in \mathbb{R}$ and eigenvectors $v_1, \ldots v_N \in \mathbb{R}^N$ such that

$$Av_n = \lambda_n v_n, \qquad n = 1, \ldots, N.$$

The set $\{v_1, \ldots, v_N\}$ is an orthonormal basis of $\mathbb{R}^N$. We call $\sigma(A) := \{\lambda_1, \ldots, \lambda_N\}$ the *spectrum* of A.

Moreover, we have the following properties:

1. A has the spectral decomposition $A = QDQ^T$ where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ and $Q = (v_1| \ldots |v_N)$. The matrix Q is orthogonal, i.e., $Q^{-1} = Q^T$ and $|Qx| = |x|$ for all x. This representation is unique up to a perturbation of the eigenvalues (or of the columns of Q).

   Proof:

2. A is invertible if, and only if, $0 \notin \sigma(A)$

   Proof:

3. If A is invertible the $\sigma(A^{-1}) = \{1/\lambda_1, \ldots, 1/\lambda_N\}$, and the eigenbasis is the same.

   Proof:

4. $\|A\| = \max_{n=1,\ldots N} |\lambda_n|$ and $\|A^{-1}\| = 1/\min |\lambda_n|$. In particular, $\kappa(A) = \max |\lambda_n| / \min |\lambda_n|$.

   Proof:

5. $h^T Ah \geq \min \lambda_n \|h\|^2$ for all $h \in \mathbb{R}^N$. In particular, A is spd if, and only if, $\lambda_n > 0$ for all n.

   Proof:

6. A is spd if, and only if, A^{-1} is spd.

   Proof:

7. If A is positive semi-definite, then $\sqrt{A} := A^{1/2} := Q\mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_N})Q^T$ is symmetric and positive semidefinite, and satisfies $(A^{1/2})^2 = A$. (In fact, it is the unique symmetric and pos. semidefinite matrix which satisfies this.) If A Is spd then $A^{1/2}$ is spd.

   Proof: